we have, by multiplying all such inequalities from $n = 2$ to $n = n$,

$$b_n > \frac{1}{n}$$

and therefore the b series is divergent and hence also the a series.

Also solved by E. H. Clarke, R. A. Johnson, H. L. Olson, Arthur Pelletier, and the Proposer.

2758 [1919, 124]. Proposed by LEONARD RICHARDSON, University of British Columbia.

Prove that, if r be a positive integer,

$$\int_0^{\pi/2} \frac{\sin\,(2r+1)\psi}{\sin\psi}\,d\psi = \frac{\pi}{2}.$$

and

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin\psi}\,d\psi = 2\left\{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^{r-1}}{2r-1}\right\}.$$

I. Solution by Elijah Swift, University of Vermont.

We have the identity

$$\sin n\psi - \sin\,(n-2)\psi = 2\sin\psi\cos\,(n-1)\psi,$$

or

$$\sin n\psi = 2\sin\psi\cos\,(n-1)\psi + \sin\,(n-2)\psi.$$

This yields us an easy reduction formula for the two integrals. For the second,

$$\int_0^{\pi/2} \frac{\sin 2r\psi}{\sin\psi}\,d\psi = 2\int_0^{\pi/2} \cos\,(2r-1)\psi\cdot d\psi + \int_0^{\pi/2} \frac{\sin\,(2r-2)\psi}{\sin\psi}\,d\psi$$

$$= 2\,\frac{(-1)^{r-1}}{2r-1} + \int_0^{\pi/2} \frac{\sin\,(2r-2)\psi}{\sin\psi}\,d\psi.$$

Applying this r times, we reach the indicated result.

For the first integral,

$$\int_0^{\pi/2} \frac{\sin\,(2r+1)\psi}{\sin\psi}\,d\psi = 2\int^{\pi/2} \cos 2r\psi\cdot d\psi + \int_0^{\pi/2} \frac{\sin\,(2r-1)\psi}{\sin\psi}\,d\psi.$$

The first of these is 0. Repeating this process r times, the required integral

$$= \int_0^{\pi/2} \frac{\sin\psi}{\sin\psi}\cdot d\psi = \frac{\pi}{2}.$$

Solved similarly by A. M. Harding and C. C. Yen.

II. Solution by R. D. Bohannan, Ohio State University.

Let $\cos\psi + i\sin\psi = z$ and $\cos\psi - i\sin\psi = 1/z$. Then, for the first integral, we have

$$\frac{\sin\,(2r+1)\psi}{\sin\psi}\,d\psi = \frac{z^{2r+1} - \dfrac{1}{z^{2r+1}}}{z - \dfrac{1}{z}}\,\frac{dz}{iz},$$

$$= \left\{z^{2r} + z^{2r-2} + \cdots + z^2 + 1 + \frac{1}{z^2} + \cdots + \frac{1}{z^{2r}}\right\}\frac{dz}{iz}.$$

The first and last terms, after integration, give

$$\frac{1}{2ri}\left(z^{2r} - \frac{1}{z^{2r}}\right) \qquad \text{or,} \qquad \frac{1}{r}\sin 2r\psi\,\Big]_0^{\pi/2},$$

which is zero; likewise, all other pairs, equidistant from the ends. The middle term gives $(\log z)/i$ or ψ, since $z = e^{i\psi}$, and with the given limits, this reduces to $\pi/2$.

Similarly, the second integral comes from

$$\left\{ z^{2r-1} + z^{2r-3} + \cdots z + \frac{1}{z} + \frac{1}{z^3} + \cdots + \frac{1}{z^{2r-1}} \right\} \frac{dz}{iz}.$$

After integration, the first and last terms, give

$$\frac{1}{i(2r-1)} \left\{ z^{2r-1} - \frac{1}{z^{2r-1}} \right\} \quad \text{or} \quad \frac{2}{2r-1} \sin (2r-1)\psi \Big]_0^{\pi/2} = \pm \frac{2}{2r-1}.$$

The result given is obtained by starting with the central terms,

$$\left(z + \frac{1}{z} \right) \frac{dz}{iz}.$$

Also solved similarly by P. J. DA CUNHA, WILLIAM HERBERG, and H. L. OLSON.

2759 [1919, 124]. Proposed by J. L. RILEY, Stephenville, Texas.

Solve the simultaneous functional equations

$$\phi(x+y) = \phi(x) + \frac{\phi(y)\cdot\psi(x)}{1 - \phi(x)\phi(y)},$$

$$\psi(x+y) = \frac{\psi(x)\cdot\psi(y)}{1 - \phi(x)\phi(y)}.$$

I. SOLUTION BY C. F. GUMMER, Kingston, Ont.

When $y = 0$ the equations show that either $\psi(x) = 0$, or $\phi(0) = 0$ and $\psi(0) = 1$. In the former case, $\phi(x)$ is constant. In the latter, the first equation gives

$$\frac{\phi(x+y) - \phi(x)}{y} = \frac{\phi(y) - \phi(0)}{y} \cdot \frac{\psi(x)}{1 - \phi(x)\phi(y)};$$

so that, if we assume that $\phi'(0)$ exists and equals a, we deduce that

$$\phi'(x) = a\psi(x).$$

If further $\psi'(0)$ exists and equals b, the second equation gives

$$\psi'(x) = \{a\phi(x) + b\}\psi(x).$$

Eliminating $\psi(x)$ between the last two equations, and writing $X(x) = a\phi(x) + b$, we get

$$X''(x) = X(x)X'(x);$$

and on integration, since $X(0) = b$ and $X'(0) = a^2$,

$$X(x) = 2A \tan (Ax + B),$$

where $\sqrt{2a^2 - b^2} = 2A$ and $b = 2A \tan B$.

Hence

$$\phi(x) = \sqrt{2} \cdot \frac{\sin Ax}{\cos (Ax + B)}, \qquad \psi(x) = \frac{\cos^2 B}{\cos^2 (Ax + B)}.$$

This solution, however, fails to satisfy the functional equations unless $A = 0$. Hence the only solutions admitting derivatives for $x = 0$ are

$$\phi(x) = c, \quad \psi(x) = 0 \qquad \text{and} \qquad \phi(x) = 0, \quad \psi(x) = 1.$$

It is apparent that $\phi(x) = 0$, $\psi(x) = a^x$ is a solution.

Possibly the proposer intended to square the denominator on the right of the second equation. With this change and similar work we get, assuming $\phi'(0)$ and $\psi'(0)$ to exist, *either*

$$\phi(x) = c, \qquad \psi(x) = 0$$